Eur. Phys. J. B **36**, 359–363 (2003)
DOI: 10.1140/epjb/e2003-00354-5

THE EUROPEAN PHYSICAL JOURNAL B

PT-invariant Helmholtz optics and its applications to slab waveguides

S.K. Moayedi[1] and A. Rostami[2,3,a]

[1] Department of Physics, Arak University, Arak, Iran
[2] Faculty of Electrical Engineering, Tabriz University, Tabriz 51664, Iran
[3] OIC Design lab., Faculty of Electrical Engineering, Tabriz University, Tabriz, Iran

Received 11 January 2003 / Received in final form 6 October 2003
Published online 23 December 2003 –

Abstract. In optical Integrated circuits (OIC) and optical device design (passive and active), the index of refraction profile is a coordinate dependent parameter. Specifically in slab waveguides which is a basic element in optical engineering, the guided modes and its behavior with respect to waveguide parameters can be extracted from the exact solution of the Helmholtz equation. In the general case, the numerical approach is applied for determination of the electric and magnetic fields. In this paper a suitable algorithm, using the PT-symmetric quantum mechanical approach has been given for light transmission through one-dimensional inhomogeneous media (PT-symmetric index of refraction).

PACS. 03.65.-w Quantum mechanics – 03.50.De Classical electromagnetism, Maxwell equations – 42.82.Et Waveguides, couplers, and arrays

1 Introduction

Numerous advances have been made in the electromagnetic field theory in recent years. This is, in part, due to new applications of the theory to many practical problems. For example, in microwave and millimeter wave applications, there is an interesting need to investigate the electromagnetic problem of new guiding structures, phase array, microwave imaging, polarimetric radar, microwave hazards, frequency sensitive surfaces, composite materials, and microwave remote sensing [1]. In the field of optics and photonics, applications involving fiber optics, integrated optical circuits, atmospheric optics, light diffusion in tissues, multi-layer periodic and non-periodic media, and optics for inhomogeneous media are among many problems whose solutions require the use of electromagnetic theory as an essential element [1]. Guided waves in inhomogeneous planar optical Waveguides such as gratings in OIC and index modulation using electro-optics and acousto-optical phenomenon in optical engineering have received considerable attention owing to their ability to perform optical signal processing and optical computing [2]. For this purpose, the characteristic of the above mentioned Waveguide with a kerr-like film layer have also been extensively investigated for ultra-high speed optical signal processing [1,2]. Some published papers so far, are based on numerical methods [3]. In an analytical approach, the supersymmetric methods were used for obtaining the indexes of refraction in inhomogeneous media [4]. Also, different analytical methods for light transmission through optical waveguides having inhomogeneous index of refraction profiles were discussed [5]. Nowadays, optical integrated circuit design and implementation is a very important subject for science and technology. The slab waveguides and also channel waveguides are basic elements for OIC. Usually, the index of refraction profiles in these Waveguides made by standard planar technology methods, is coordinate dependent. The exact determination of the index of refraction profile needs to solve the diffusion equations. The constant parameters in this equation depend on the process variable such as temperature. In general exact solution for the diffusion equation can not be found. In practice, by applying some assumption, the Gaussian profile or error function profile can be used. But, in the general case, the numerical approach is applied for solving the diffusion equation and obtaining the index of refraction profile (Graded Index Profile). So, in most of applications the index of refraction can be approximated by one or more suitable polynomial functions. By this approximation, we must solve the Helmholtz equation with a polynomial function for the index of refraction. In this paper we will present the exact solution for these cases. For this aim, the PT-symmetric index of refractions are introduced and exact solutions for TE-Mode of electromagnetic field in a slab waveguide in terms of well known physical functions are presented. The organization of this paper is as follows. In Section 2, the concept of PT-symmetric

[a] e-mail: rostami@tabrizu.ac.ir

index of refraction is introduced. Also, the analytical solution of Helmholtz equation is presented. Our analytical and simulated results based on the PT-symmetric idea for anharmonic index of refraction profiles are discussed in Section 3. Finally, the paper ends with a conclusion.

2 PT-symmetric index of refraction

In this section, we will obtain the exactly solvable index of refraction profiles based on the PT-symmetric concept in quantum mechanics [6,7]. First, we present the Helmholtz wave equation for TE-Mode of electromagnetic field in waveguides as shown in Figure 1.

Let us consider a plane wave incident upon a medium whose dielectric constant is a function of height X. We choose the Z-axis specially such that the plane of incidence is in the X-Z plane. This is a two dimensional problem $\left(\frac{\partial}{\partial Y}=0\right)$, and thus there are two independent TE and TM waves. These two modes are equivalent and so we can apply the method to the TE mode only. By considering the time Harmonic dependency $(E_Y(X,Z,t)=E_Y(X,Z)e^{i\omega t})$ for the electric and magnetic fields, we obtain the following forms for the Maxwell equations

$$\begin{aligned}\nabla\times E &= -i\,\omega\,\mu\,H\\ \nabla\times H &= i\,\omega\,\varepsilon\,E.\end{aligned}\tag{2.1}$$

Using equation (2.1), we have ($\mu=$ const., $\varepsilon=\varepsilon(X)$)

$$\nabla\times\nabla\times E=\nabla(\nabla E)-\nabla^2E=\omega^2\mu\,\varepsilon\,E.$$

Noting that for a dielectric media in the absence of any charge density anywhere ($\nabla D=0$) and $\frac{\partial E_Y}{\partial Y}=0$, we get

$$\left[\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Z^2}+\omega^2\mu\,\varepsilon\,(X)\right]E_Y(X,Z)=0\tag{2.2}$$

where $\omega^2\mu\varepsilon(X)=k_0^2n^2(X)=k_0^2n_0^2+k_0^2n_1g(X)$ and n_1g is a dimensionless quantity. Also, let us consider a TE-plane wave obliquely incident upon the medium shown in Figure 1. So, we can write

$$E_Y(X,Z)=E_Y(X)e^{-ikZ}.\tag{2.3}$$

Then, the Helmholtz equation for light transmission in this Waveguide is

$$\left[\frac{d^2}{dX^2}+k_0^2n_1g(X)\right]E_Y(X)=(k^2-k_0^2n_0^2)E_Y(X).\tag{2.4}$$

Here, we want to determine the polynomial type of the index of refraction in which equation (2.4) has exact solutions. Now, we define the new dimensionless variable x as

$$x=k_0X.\tag{2.5}$$

In this equation x is dimensionless and X is the actual length. Using this newly introduced dimensionless variable, equation (2.4) can be converted to

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}-\frac{1}{2}n_1g(x)\right]E_Y(x)=-\frac{1}{2}\left(\frac{k^2}{k_0^2}-n_0^2\right)E_Y(x).\tag{2.6}$$

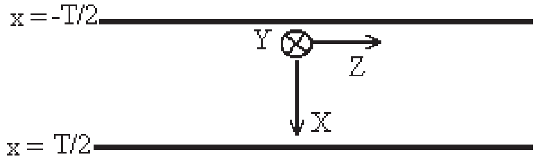


Fig. 1. Slab waveguide structure.

Now, we apply the PT-symmetric quantum mechanical methods to solve the Helmholtz equation (Eq. (2.6)). Until 1998 [6–8], the Hermiticity of the Hamiltonian was supposed to be the necessary condition for having real spectrum. A conjecture due to Bender and Bottcher, has relaxed this condition in a very inspiring way by introducing the concept of PT-symmetric Hamiltonians. Here, P denotes the parity operation (space reflection) $(x\to -x)$ and T mimics the time reversal $(i\to -i)$. If $(PT)H(PT)^{-1}=H$ and if $(PT)\Psi(x)=\pm\Psi(x)$, the energy will be real and have complex conjugate pairs if the latter consideration is not satisfied [7]. So, using this idea, several PT-symmetric index of refraction appeared to have real wave vectors. Now, using the analogy between Helmholtz wave equation (Eq. (2.6)) and the Schrödinger like equation [5], we obtain the following relations

$$\begin{aligned}-\frac{1}{2}n_1g(x)&=V(x)\\ E&=\frac{n_0^2-\dfrac{k^2}{k_0^2}}{2}.\end{aligned}\tag{2.7}$$

The quantity E in equation (2.7) is only a dimensionless eigenvalue. Now, let us consider the following anharmonic index of refraction

$$-\frac{1}{2}n_1g(x)=\sum_{i=1}^{6}C_ix^i\tag{2.8}$$

where the C_i's are dimensionless constants. In order to obtain a PT symmetric invariant index of refraction $n^2(x)=n_0^2+n_1g(x)$, the coefficients $C_{1,3,5}$ and $C_{2,4,6}$ must be pure imaginary and real constants respectively in equation (2.8). Now, we consider [6] the exact solution for equation (2.6) as $E_Y(x)=E_0f(x)e^{-\sum_{j=1}^{4}b_jx^j}$, where E_0 is a real constant with the dimensions of the electric field, $f(x)$ is some dimensionless function of variable x and b_i's are dimensionless constants. In this work, we concentrate on three choices of $f(x)$ as follows

a) $f(x)=1$
b) $f(x)=x+a_0$
c) $f(x)=x^2+a_1x+a_0$

where a_0 and a_1 are dimensionless constants. According to our knowledge from waveguide theory, the different $f(x)$ mentioned above correspond to different waveguide modes.The details of the calculations for the three cases mentioned above are given in the next section.

Table 1. Final results for $f(x) = 1, f(x) = x + a_0, f(x) = x^2 + a_1x + a_0$.

$f(x) = 1$	$k^2 = k_0^2(n_0^2 - 2b_2)$
	$n^2(x) = n_0^2 - (x^6 + 4b_2x^4 + (4b_2^2 - 3)x^2)$ $E_Y(x) = E_0e^{-\frac{1}{4}x^4 - b_2x^2}$
$f(x) = x + a_0$	$k^2 = k_0^2(n_0^2 - 6b_2)$
	$n^2(x) = n_0^2 - (x^6 + 4b_2x^4 + (4b_2^2 - 5)x^2)$ $E_Y(x) = E_0xe^{-\frac{1}{4}x^4 - b_2x^2}$
$f(x) = x^2 + a_1x + a_0$	$k_\pm^2 = k_0^2\left(n_0^2 - 6b_2 \mp 2\sqrt{4b_2^2 + 2}\right)$
	$n^2(x) = n_0^2 - (x^6 + 4b_2x^4 + (4b_2^2 - 7)x^2)$ $E_Y^\pm(x) = E_0\left[x^2 + b_2 \mp \sqrt{b_2^2 + \frac{1}{2}}\right]e^{-\frac{1}{4}x^4 - b_2x^2}$

3 Results and discussion

Now, we give the detail of calculations for the electric field, index of refraction and wave vector in subsections *a*, *b*, *c* and *d*.

a) $f(x) = 1$
For this case the electric field in TE-Mode is given by

$$E_Y(x) = E_0e^{-\sum_{j=1}^{4} b_jx^j}. \tag{3.1}$$

After substitution of equation (3.1) in the Helmholtz equation (Eq. (2.6)) and using the equation (2.8), we have

$$\begin{aligned} C_1 &= -3b_3 + 2b_1b_2 \\ C_2 &= -6b_4 + 3b_1b_3 + 2b_2^2 \\ C_3 &= 4b_1b_4 + 6b_2b_3 \\ C_4 &= 8b_2b_4 + \frac{9}{2}b_3^2 \\ C_5 &= 12b_3b_4 \\ C_6 &= 8b_4^2 \\ k^2 &= k_0^2(n_0^2 - 2b_2 + b_1^2). \end{aligned} \tag{3.2}$$

In order to obtain a real index of refraction, we assume that $b_1 = b_3 = 0$ in equation (3.2), hence we have

$$\begin{aligned} C_1 &= C_3 = C_5 = 0 \\ C_2 &= -6b_4 + 2b_2^2 \\ C_4 &= 8b_2b_4 \\ C_6 &= 8b_4^2 \\ k^2 &= k_0^2(n_0^2 - 2b_2). \end{aligned} \tag{3.3}$$

Then, the electric field in equation (3.1) is

$$E_Y(x) = E_0e^{-b_4x^4 - b_2x^2}. \tag{3.4}$$

To have a finite electric field at infinity ($x \to \pm\infty$), b_4 must be a real positive constant. By considering $C_6 = \frac{1}{2}$ in equation (3.3), we obtain

$$\begin{aligned} b_4 &= \frac{1}{4} \\ C_2 &= -\frac{3}{2} + 2b_2^2 \\ C_4 &= 2b_2. \end{aligned} \tag{3.5}$$

The final results for the above example is given in Table 1. The simulated results for the electric field and the index of refraction profiles for $f(x) = 1$ are shown in Figure 2.

b) $f(x) = x + a_0$
For this case the electric field in TE-Mode is given by

$$E_Y(x) = E_0(x + a_0)e^{-\frac{1}{4}x^4 - b_3x^3 - b_2x^2 - b_1x}. \tag{3.6}$$

After substitution of equation (3.6) in the Helmholtz equation (Eq. (2.6)) and using equation (2.8), we obtain the following relations

$$\begin{aligned} C_1 &= -6b_3 + 2b_1b_2 + a_0 \\ C_2 &= -\frac{5}{2} + 3b_1b_3 + 2b_2^2 \\ C_3 &= b_1 + 6b_2b_3 \\ C_4 &= 2b_2 + \frac{9}{2}b_3^2 \\ C_5 &= 3b_3 \\ C_6 &= \frac{1}{2} \\ a_0^3 &- 3b_3a_0^2 + 2b_2a_0 - b_1 = 0. \end{aligned} \tag{3.7}$$

In this case the wave vector is given by

$$k^2 = k_0^2(n_0^2 + b_1^2 - 6b_2 + 6a_0b_3 - 2a_0^2). \tag{3.8}$$

As a special example, let us assume that

$$b_1 = b_3 = a_0 = 0.$$

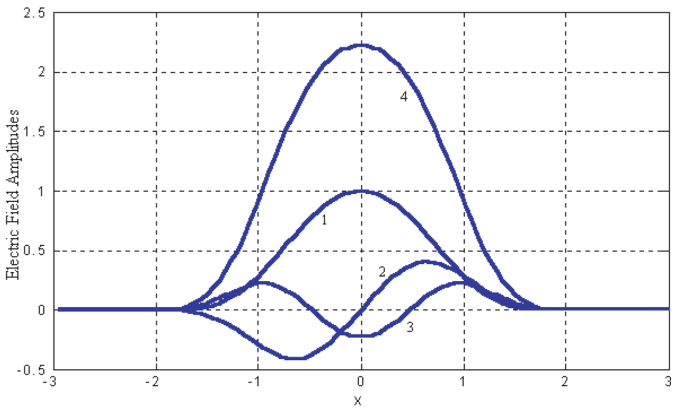
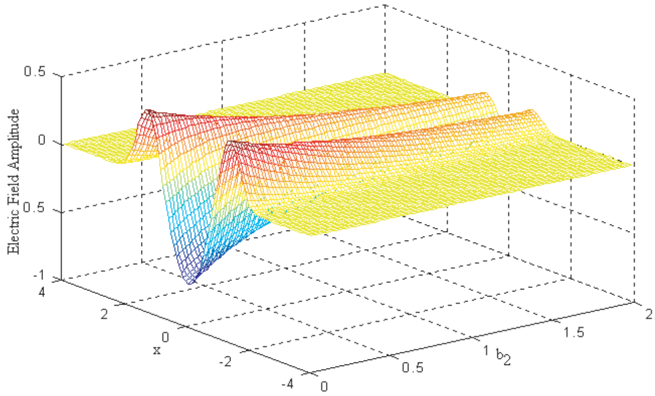
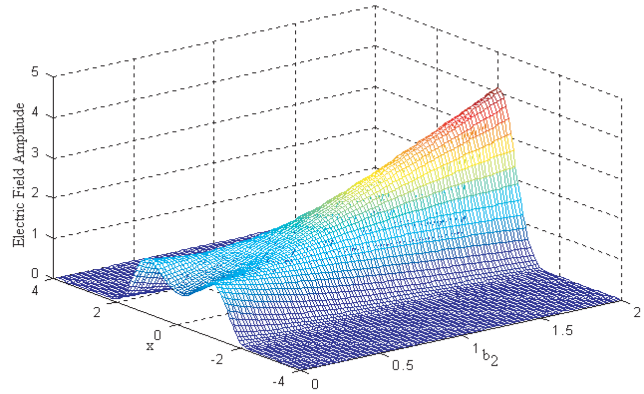


Fig. 2. Electric field amplitude vs. $x\,(b_2 = 1,\ E_0 = 1)$; 1: E_Y for case a; 2: E_Y for case b; 3: E_Y for case c (E_Y^-); 4: E_Y for case c (E_Y^+).



(a)



(b)

Fig. 3. (a) Electric field amplitude for case c (E_Y^+) vs. b_2 and $x\,(E_0 = 1)$. (b) Electric field amplitude for case c (E_Y^-) vs. b_2 and $x\,(E_0 = 1)$.

So, by using equations (3, 7, 8), we obtain

$$\begin{aligned} &C_1 = C_3 = C_5 = 0 \\ &C_2 = -\frac{5}{2} + 2b_2^2 \\ &C_4 = 2b_2 \\ &C_6 = \frac{1}{2} \\ &k^2 = k_0^2(n_0^2 - 6b_2). \end{aligned} \tag{3.9}$$

The corresponding electric field is

$$E_Y(x) = E_0 x e^{-\frac{1}{4}x^4 - b_2x^2}. \tag{3.10}$$

The final results for this case is given in Table 1. Also, the simulated results are given in Figure 2.

c) $f(x) = x^2 + a_1x + a_0$
If we perform as in the two previous cases, we have the following results for $b_3 = 0$,

$$\begin{aligned} E_Y^{(\pm)}(x) &= E_0\left[x^2 + b_2 \mp \sqrt{b_2^2 + \frac{1}{2}}\right] e^{-\frac{1}{4}x^4 - b_2x^2} \\ k_\pm^2 &= k_0^2\left(n_0^2 - 6b_2 \mp 2\sqrt{4b_2^2 + 2}\right). \end{aligned} \tag{3.11}$$

The final results for this case is given in Table 1. The simulated results are given in Figures (2–6).

d) Here, we are interested in examples with complex indexes of refraction. By the special choice

$$b_1 = i, \quad b_3 = 0, \quad b_4 = \frac{1}{4},$$

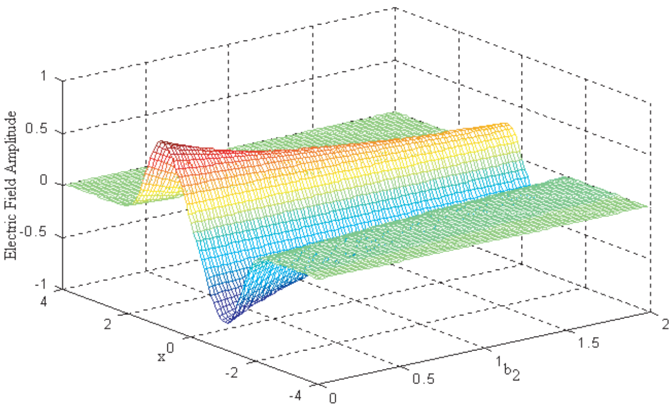


Fig. 4. Electric field amplitude for case b vs. b_2 and $x\,(E_0 = 1)$.

the coefficients C_i's in equation (3.2) are

$$\begin{aligned} &C_1 = 2b_2 i \\ &C_2 = 2b_2^2 - \frac{3}{2} \\ &C_3 = i \\ &C_4 = 2b_2 \\ &C_5 = 0 \\ &C_6 = \frac{1}{2}. \end{aligned} \tag{3.12}$$

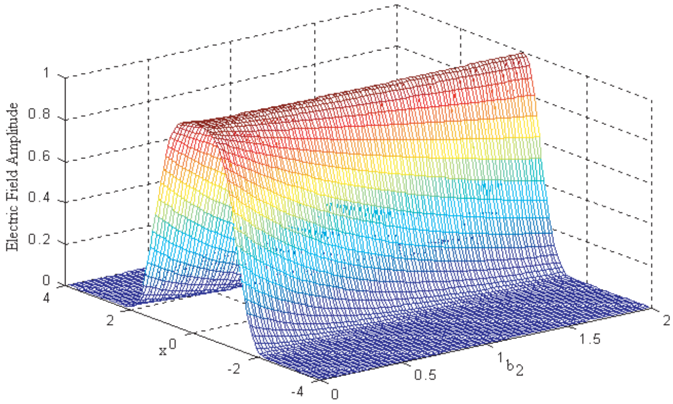


Fig. 5. Electric field amplitude for case a vs. b_2 and x ($E_0 = 1$).

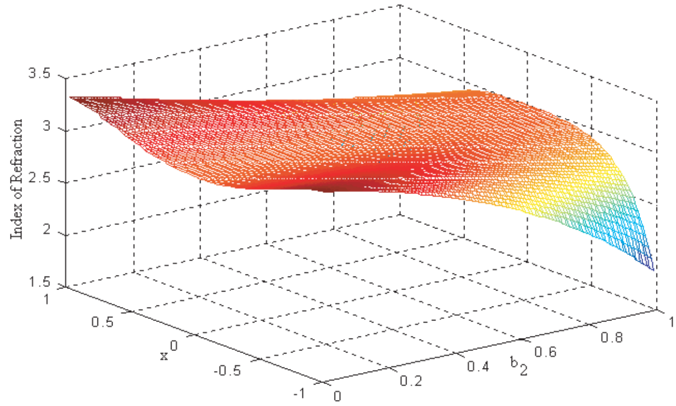


Fig. 6. Index of refraction profile for case a vs. b_2 and x ($n_0 = 3$).

The corresponding explicit forms for the wave vector, electric field and complex index of refraction are

$$\begin{aligned} k^2 &= k_0^2(n_0^2 - 2b_2 - 1) \\ E_Y(x) &= E_0 e^{-\frac{1}{4}x^4 - b_2 x^2 - ix} \\ n^2(x) &= n_0^2 - \big[x^6 + 4b_2 x^4 + (4b_2^2 - 3)x^2 \\ &\quad + i(2x^3 + 4b_2 x)\big]. \end{aligned} \tag{3.13}$$

As a second example for a media with complex index of refraction, we choose

$$a_0 = 0, \quad b_1 = 0, \quad b_3 = \frac{i}{3},$$

the coefficients C_i's in equation (3.7) are

$$\begin{aligned} C_1 &= -2i \\ C_2 &= 2b_2^2 - \frac{5}{2} \\ C_3 &= 2b_2 i \\ C_4 &= 2b_2 - \frac{1}{2} \\ C_5 &= i \\ C_6 &= \frac{1}{2}. \end{aligned} \tag{3.14}$$

In this case the explicit forms for the wave vector, electric field and complex index of refraction are

$$\begin{aligned} k^2 &= k_0^2(n_0^2 - 6b_2) \\ E_Y(x) &= E_0 x e^{-\frac{1}{4}x^4 - \frac{i}{3}x^3 - b_2 x^2} \\ n^2(x) &= n_0^2 - \big[x^6 + (4b_2 - 1)x^4 + (4b_2^2 - 5)x^2 \\ &\quad + i(2x^5 + 4b_2 x^3 - 4x)\big]. \end{aligned} \tag{3.15}$$

It is important to note that the indexes of refraction appear in equations (3.13,15) have the following property $(PT)n^2(x)(PT)^{-1} = n^2(x)$ (i.e. They are PT-invariant index of refractions).

4 Conclusion

In this paper, we have proposed a new set of the indexes of refraction in which the Helmholtz equation has exact solutions. In this work, the PT-symmetric idea has been used for this proposal. Some exact solutions for the electromagnetic fields for our proposed PT-Symmetric index of refraction are presented. Our proposal for the index of refraction is practical and can be implemented easily with molecular beam epitaxy (MBE) and metal organic chemical vapour deposition (MOCVD) which are standard planar integrated circuit technologies.

References

1. A. Ishimaru, *Electromagnetic Wave Propagation, Radiation and Scattering* (Prentice Hall, 1991)
2. A. Yariv, *Quantum Electronics*, 3rd (John Wiley, 1989)
3. Li-Pen Yuan, IEEE J. QE **30**, 1 (1994)
4. S.M. Chumakov, K.B. Wolf, Phys. Lett. A **193**, 51 (1994)
5. A. Rostami, S.K. Moayedi, Indian J. Phys. B **75**, 357 (2001), A. Rostami, S.K. Moayedi, J. Opt. A: Pure Appl. Opt. **5**, 380 (2003), and A. Rostami, Laser Physics, Vol. **12**, 1358 (2002)
6. B. Bagchi, F. Cannata, C. Quesne, Phys. Lett. A **269**, 79 (2000)
7. Z. Ahmed, `LANL Archives: quant-ph/0108016 V2 10` Aug. (2001)
8. C.M. Bender, S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998)